

On the algebraic set of singular elements in a complex simple Lie algebra

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Abstract. Let G be a complex simple Lie group and let $\mathfrak{g} = \text{Lie } G$. Let $S(\mathfrak{g})$ be the G -module of polynomial functions on \mathfrak{g} and let $\text{Sing } \mathfrak{g}$ be the closed algebraic cone of singular elements in \mathfrak{g} . Let $\mathcal{L} \subset S(\mathfrak{g})$ be the (graded) ideal defining $\text{Sing } \mathfrak{g}$ and let $2r$ be the dimension of a G -orbit of a regular element in \mathfrak{g} . Then $\mathcal{L}^k = 0$ for any $k < r$. On the other hand, there exists a remarkable G -module $M \subset \mathcal{L}^r$ which already defines $\text{Sing } \mathfrak{g}$. The main results of this paper are a determination of the structure of M .

0. Introduction

0.1. Let G be a complex simple Lie group and let $\mathfrak{g} = \text{Lie } G$. Let $\ell = \text{rank } \mathfrak{g}$. Then in superscript centralizer notation one has $\dim \mathfrak{g}^x \geq \ell$ for any $x \in \mathfrak{g}$. An element $x \in \mathfrak{g}$ is called regular (resp. singular) if $\dim \mathfrak{g}^x = \ell$ (resp. $> \ell$). Let $\text{Reg } \mathfrak{g}$ be the set of all regular elements in \mathfrak{g} and let $\text{Sing } \mathfrak{g}$, its complement in \mathfrak{g} , be the set of all singular elements in \mathfrak{g} . Then one knows that $\text{Reg } \mathfrak{g}$ is a nonempty Zariski open subset of \mathfrak{g} and hence $\text{Sing } \mathfrak{g}$ is a closed proper algebraic subset of \mathfrak{g} .

Let $S(\mathfrak{g})$ (resp. $\wedge \mathfrak{g}$) be the symmetric (resp. exterior) algebra over \mathfrak{g} . Both algebras are graded and are G -modules by extension of the adjoint representation. Let \mathcal{B} be the natural extension of the Killing form to $S(\mathfrak{g})$ and $\wedge \mathfrak{g}$. The inner product it induces on u and v in either $S(\mathfrak{g})$ or $\wedge \mathfrak{g}$ is denoted by (u, v) . The use of \mathcal{B} permits an identification of $S(\mathfrak{g})$ with the algebra of polynomial functions on \mathfrak{g} . Since $\text{Sing } \mathfrak{g}$ is clearly a cone the ideal, \mathcal{L} , of all $f \in S(\mathfrak{g})$ which vanish on $\text{Sing } \mathfrak{g}$ is graded. Let $n = \dim \mathfrak{g}$ and let $r = (n - \ell)/2$. One knows that $n - \ell$ is even so that $r \in \mathbb{Z}_+$. It is easy to show that

$$\mathcal{L}^k = 0, \text{ for all } k < r. \quad (0.1)$$

The purpose of this paper is to define and study a rather remarkable G -submodule

$$M \subset \mathcal{L}^r \quad (0.2)$$

which in fact defines $\text{Sing } \mathfrak{g}$. That is, if $x \in \mathfrak{g}$, then

$$x \in \text{Sing } \mathfrak{g} \iff f(x) = 0, \quad \forall f \in M \quad (0.3)$$

0.2. We will now give a definition of M . The use of \mathcal{B} permits an identification of $\wedge \mathfrak{g}$ with the underlying space of the cochain complex defining the cohomology of \mathfrak{g} . The coboundary operator is denoted here by d (and δ in [Kz]) is a (super) derivation of degree 1 of $\wedge \mathfrak{g}$ so that $dx \in \wedge^2 \mathfrak{g}$ for any $x \in \mathfrak{g}$. Since $\wedge^{\text{even}} \mathfrak{g}$ is a commutative algebra there exists a homomorphism

$$\gamma : S(\mathfrak{g}) \rightarrow \wedge^{\text{even}} \mathfrak{g}$$

where for $x \in \mathfrak{g}$, $\gamma(x) = -dx$. One readily has that

$$S^k(\mathfrak{g}) \subset \text{Ker } \gamma, \text{ for all } k > r. \quad (0.4)$$

Let $\gamma_r = \gamma|_{S^r(\mathfrak{g})}$ so that

$$\gamma_r : S^r(\mathfrak{g}) \rightarrow \wedge^{2r} \mathfrak{g}. \quad (0.5)$$

If $x \in \mathfrak{g}$, one readily has

$$x^r \in \text{Ker } \gamma_r \iff x \in \text{Sing } \mathfrak{g}. \quad (0.6)$$

Let Γ be the transpose of γ_r so that one has a G -map

$$\Gamma : \wedge^{2r} \rightarrow S^r(\mathfrak{g}). \quad (0.7)$$

By definition

$$M = \text{Im } \Gamma. \quad (0.8)$$

0.3. Let $J = S(\mathfrak{g})^G$ so that (Chevalley) J is a polynomial ring $\mathbb{C}[p_1, \dots, p_\ell]$ where the invariants p_j can be chosen to be homogeneous. In fact if $m_j, j = 1, \dots, \ell$, are the exponents of \mathfrak{g} we can take $\deg p_j = m_j + 1$. For any linearly independent $u_1, \dots, u_\ell \in \mathfrak{g}$, let

$$\psi(u_1, \dots, u_\ell) = \det \partial_{u_i} p_j \quad (0.9)$$

where, if $v \in \mathfrak{g}$, ∂_v is the operator of partial derivative by v in $S(\mathfrak{g})$. One has

$$\psi(u_1, \dots, u_\ell) \in S^r(\mathfrak{g}) \quad (0.10)$$

since, as one knows, $\sum_{i=1}^\ell m_i = r$.

Let Σ_{2r} be the permutation group of $\{1, \dots, 2r\}$ and let $\Pi_r \subset \Sigma_{2r}$ be a subset (of cardinality $(2r-1)(2r-3)\cdots 1$) with the property that $sg\nu = 1$ for all $\nu \in \Pi_r$ and such that, as unordered,

$$\{(\nu(1), \nu(2)), \dots, (\nu(2r-1), \nu(2r))\} \mid \nu \in \Pi_r\}$$

is the set of all partitions of $\{1, \dots, 2r\}$ into a union of r subsets each of which has two elements. The following is one of our main theorems. Even more than explicitly determining $\psi(u_1, \dots, u_\ell)$ one has

Theorem 0.1. *Let u_1, \dots, u_ℓ be any ℓ linearly independent elements in \mathfrak{g} and let w_1, \dots, w_{2r} be a basis of the \mathcal{B} -orthogonal subspace to the span of the u_i . Then there exists some fixed $\kappa \in \mathbb{C}^\times$ such that, for all $x \in \mathfrak{g}$,*

$$\sum_{\nu \in \Pi_r} ([w_{\nu(1)}, w_{\nu(2)}], x) \cdots ([w_{\nu(2r-1)}, w_{\nu(2r)}], x) = \kappa \psi(u_1, \dots, u_\ell)(x). \quad (0.11)$$

Moreover $\psi(u_1, \dots, u_\ell) \in M$. In fact the left side of (0.11) is just $\Gamma(w_1 \wedge \cdots \wedge w_{2r})(x)$. In addition M is the span of $\psi(u_1, \dots, u_\ell)$, over all $\{u_1, \dots, u_\ell\}$, taken from the $\binom{n}{\ell}$ subsets of ℓ -elements in any given basis of \mathfrak{g} .

We now deal with the G -module structure of M . For any subspace \mathfrak{s} of \mathfrak{g} , say of dimension k , let $[\mathfrak{s}] = \mathbb{C}v_1 \wedge \cdots \wedge v_k \subset \wedge^k \mathfrak{g}$ where the v_i are a basis of \mathfrak{s} . Let

\mathfrak{h} be a Cartan subalgebra of \mathfrak{g} and let Δ be the set of roots for the pair $(\mathfrak{h}, \mathfrak{g})$. For any $\varphi \in \Delta$ let $e_\varphi \in \mathfrak{g}$ be a corresponding root vector. Let $\Delta_+ \subset \Delta$ be a choice of a set of positive roots and let \mathfrak{b} be the Borel subalgebra spanned by \mathfrak{h} and all e_φ for $\varphi \in \Delta_+$. For any subset $\Phi \subset \Delta$ let $\mathfrak{a}_\Phi \subset \mathfrak{g}$ be the span of e_φ for $\varphi \in \Phi$. Also let $\langle \Phi \rangle = \sum_{\varphi \in \Phi} \varphi$ so that

$$[\mathfrak{a}_\Phi] \text{ is an } \mathfrak{h}\text{-weight space for the } \mathfrak{h}\text{-weight } \langle \Phi \rangle. \quad (0.12)$$

A subset $\Phi \in \Delta_+$ will be said to be an ideal in Δ_+ if \mathfrak{a}_Φ is an ideal of \mathfrak{b} . In such a case, if $\text{card } \Phi = k$, then the span V_Φ of $G \cdot [a_\Phi]$ is an irreducible G -submodule of $\wedge^k \mathfrak{g}$ having $[a_\Phi]$ as highest weight space and $\langle \Phi \rangle$ as highest weight. Let \mathcal{I} be the set of all ideals Φ in Δ_+ of cardinality ℓ . It is shown in [KW] that all ideals in \mathfrak{b} of dimension ℓ are abelian and hence are of the form \mathfrak{a}_Φ for a unique $\Phi \in \mathcal{I}$. Specializing k in [K3] to ℓ one has that, by definition, $A_\ell \subset \wedge^\ell \mathfrak{g}$ is the span of $[\mathfrak{s}]$ over all abelian subalgebras $\mathfrak{s} \subset \mathfrak{g}$ of dimension ℓ . Using results in [K3] and that in [KW] above, one also has that A_ℓ is a multiplicity one G -module with the complete reduction

$$A_\ell = \oplus_{\Phi \in \mathcal{I}} V_\Phi \quad (0.13)$$

so that there are exactly $\text{card } \mathcal{I}$ irreducible components. In addition it has been shown in [K3] that ℓ is the maximal eigenvalue of the (\mathcal{B} normalized) Casimir operator, Cas , in $\wedge^\ell \mathfrak{g}$ and A_ℓ is the corresponding eigenspace. In the present paper the G -module structure of M is given in

Theorem 0.2. *As G -modules one has an equivalence*

$$M \cong A_\ell \quad (0.14)$$

so that M is a multiplicity one module with $\text{card } \mathcal{I}$ irreducible components. Moreover the components can be parameterized by \mathcal{I} in such a way that the component corresponding to $\Phi \in \mathcal{I}$ has highest weight $\langle \Phi \rangle$. In addition Cas takes the value ℓ on each and every irreducible component of M .

1. Preliminaries

1.1. Let \mathfrak{g} be a complex semisimple Lie algebra and let G be a Lie group such that $\mathfrak{g} = \text{Lie } G$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of \mathfrak{g} and let ℓ be the rank of \mathfrak{g} so that $\ell = \dim \mathfrak{h}$. Let Δ be the set of roots for the pair $(\mathfrak{h}, \mathfrak{g})$ and let $\Delta_+ \subset \Delta$ be a choice of a set of positive roots. Let $r = \text{card } \Delta_+$ so that

$$n = \ell + 2r \tag{1.1}$$

where we let $n = \dim \mathfrak{g}$. Let \mathcal{B} be Killing form (x, y) on \mathfrak{g} . For notational economy we identify \mathfrak{g} with its dual \mathfrak{g}^* using \mathcal{B} . The bilinear form \mathcal{B} extends to an inner product (p, q) , still denoted by \mathcal{B} , on the two graded algebras, the symmetric algebra $S(\mathfrak{g})$ of \mathfrak{g} and the exterior algebra $\wedge \mathfrak{g}$ of \mathfrak{g} . If $x_i, y_j \in \mathfrak{g}$, $i = 1, \dots, k$, $j = 1, \dots, m$, then the product of x_i is orthogonal to the product of y_j in both $S(\mathfrak{g})$ and $\wedge \mathfrak{g}$ if $k \neq m$, whereas if $k = m$,

$$\begin{aligned} (x_1 \cdots x_k, y_1 \cdots y_k) &= \sum_{\sigma \in \Sigma_k} (x_1, y_{\sigma(1)}) \cdots (x_k, y_{\sigma(k)}) \quad \text{in } S(\mathfrak{g}) \\ (x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k) &= \sum_{\sigma \in \Sigma_k} sg(\sigma) (x_1, y_{\sigma(1)}) \cdots (x_k, y_{\sigma(k)}) \quad \text{in } \wedge \mathfrak{g}. \end{aligned} \tag{1.2}$$

Here Σ_k is the permutation group on $\{1, \dots, k\}$ and sg abbreviates the signum character on Σ_k .

The identification of \mathfrak{g} with its dual has the effect of identifying $S(\mathfrak{g})$ with the algebra of polynomial functions $f(y)$ on \mathfrak{g} . Thus if $x, y \in \mathfrak{g}$, then $x(y) = (x, y)$ and if $x_i \in \mathfrak{g}$, $i = 1, \dots, k$, then

$$\begin{aligned} (x_1 \cdots x_k)(y) &= \prod_{i=1}^k (x_i, y) \\ &= (x_1 \cdots x_k, \frac{1}{k!} y^k). \end{aligned} \tag{1.3}$$

The identification of \mathfrak{g} with its dual also has the effect of identifying the (supercommutative) algebra $\wedge \mathfrak{g}$ with the underlying space of the standard cochain complex

defining the cohomology of \mathfrak{g} . Let d be the (super) derivation of degree 1 of $\wedge \mathfrak{g}$, defined by putting

$$d = \frac{1}{2} \sum_{i=1}^n \varepsilon(w_i) \theta(z_i). \quad (1.4)$$

Here $\varepsilon(u)$, for any $u \in \wedge \mathfrak{g}$, is left exterior multiplication by u so that $\varepsilon(u)v = u \wedge v$ for any $v \in \wedge \mathfrak{g}$. Also $w_i, i = 1, \dots, n$, is any basis of \mathfrak{g} and $z_i \in \mathfrak{g}, i = 1, \dots, n$, is the \mathcal{B} dual basis. $\theta(x)$, for $x \in \mathfrak{g}$, is the derivation of $\wedge \mathfrak{g}$, of degree 0, defined so that $\theta(x)y = [x, y]$ for any $y \in \mathfrak{g}$. One readily notes that (1.4) is independent of the choice of the basis w_i . Thus if $x \in \mathfrak{g}$, then $dx \in \wedge^2 \mathfrak{g}$ is given by

$$dx = \frac{1}{2} \sum_{i=1}^n w_i \wedge [z_i, x]. \quad (1.5)$$

Any element $\omega \in \wedge^2 \mathfrak{g}$ defines an alternating bilinear form on \mathfrak{g} . Its value $\omega(y, z)$ on $y, z \in \mathfrak{g}$ may be given in terms of \mathcal{B} by

$$\omega(y, z) = (\omega, y \wedge z). \quad (1.6)$$

The rank of ω is necessarily even. In fact if $\text{rank } \omega = 2k$, then there exist $2k$ linearly independent elements $v_i \in \mathfrak{g}, i = 1, \dots, 2k$, such that

$$\omega = v_1 \wedge v_2 + \dots + v_{2k-1} \wedge v_{2k}. \quad (1.7)$$

The radical of ω , denoted by $\text{Rad } \omega$, is the space of all $y \in \mathfrak{g}$ such that $\omega(y, z) = 0$ for all $z \in \mathfrak{g}$. For $u \in \wedge \mathfrak{g}$, let $\iota(u)$ be the transpose of $\varepsilon(u)$ with respect to \mathcal{B} on $\wedge \mathfrak{g}$. If $u = y \in \mathfrak{g}$, then one knows that $\iota(y)$ is the (super) derivation of degree minus 1 defined so that if $z \in \mathfrak{g}$, then $\iota(y)z = (y, z)$. (See p. 8 in [Kz]). From (1.6) one has

$$\text{Rad } \omega = \{y \in \mathfrak{g} \mid \iota(y)\omega = 0\}. \quad (1.8)$$

If \mathfrak{s} is any subspace of \mathfrak{g} , let \mathfrak{s}^\perp be the \mathcal{B} orthogonal subspace to \mathfrak{s} . From (1.7) one then has that

$$\{v_i\}, i = 1, \dots, 2k, \text{ is a basis of } \text{Rad } \omega^\perp. \quad (1.9)$$

If $\mathfrak{s} \subset \mathfrak{g}$ is any subspace, say of dimension m , let $[\mathfrak{s}] \in \wedge^m \mathfrak{g}$ be the \mathbb{C} span of the decomposable element $u_1 \wedge \cdots \wedge u_m$ where $\{u_i, i = 1, \dots, m\}$ is a basis of \mathfrak{s} . One notes that if $\omega \in \wedge^2 \mathfrak{g}$ is given as in (1.7), then

$$\omega^k = k! \ v_1 \wedge \cdots \wedge v_{2k} \quad (1.10)$$

so that

$$\omega^j \neq 0 \iff j \leq k \text{ and } \omega^k \in [\text{Rad } \omega^\perp]. \quad (1.11)$$

Let $\{w_j, j = 1, \dots, n\}$ be a \mathcal{B} orthonormal basis of \mathfrak{g} . Put $\mu = w_1 \wedge \cdots \wedge w_n$ so that

$$(\mu, \mu) = 1 \quad (1.12)$$

so that μ is unique up to sign and $\wedge^n \mathfrak{g} = \mathbb{C}\mu$. For any $v \in \wedge \mathfrak{g}$ let $v^* = \iota(v)\mu$. We recall the more or less well known.

Proposition 1.1. *If $\mathfrak{s} \subset \mathfrak{g}$ is any subspace and $0 \neq u \in [\mathfrak{s}]$, then*

$$0 \neq u^* \in [\mathfrak{s}^\perp]. \quad (1.13)$$

Moreover if $s, t \in \wedge \mathfrak{g}$, one has

$$(s, t) = (s^*, t^*). \quad (1.14)$$

Proof. Let $\{y_i, i = 1, \dots, m\}$ be a basis of \mathfrak{s} chosen so that $u = y_1 \wedge \cdots \wedge y_m$ and let $\{z_j, j = 1, \dots, n - m\}$ be a basis of \mathfrak{s}^\perp . Then if $y'_k, k = 1, \dots, m$, are chosen in \mathfrak{g} such that $(y_i, y'_k) = \delta_{ik}$, it is immediate that the y'_k together with the z_j form a basis of \mathfrak{g} so that for some $\lambda \in \mathbb{C}^\times$ one has

$$\lambda y'_1 \wedge \cdots \wedge y'_m \wedge z_1 \wedge \cdots \wedge z_{n-m} = \mu. \quad (1.15)$$

But since interior product is the transpose of exterior product one has

$$\iota(q) \iota(p) = \iota(p \wedge q) \quad (1.16)$$

for any $p, q \in \wedge^{\mathfrak{g}}$. Thus by (1.15) one has

$$u^* = \lambda z_1 \wedge \cdots \wedge z_{n-m}$$

establishing (1.13). To prove (1.14) it suffices by linearity to assume that both s and t are decomposable of some degree m . Thus we can assume $s = y_1 \wedge \cdots \wedge y_m$ and $t = z_1 \wedge \cdots \wedge z_m$ for $y_i, z_j \in \mathfrak{g}$. But now, as one knows, and readily establishes,

$$\varepsilon(y) \iota(z) + \iota(z) \varepsilon(y) = (y, z) \text{Id}_{\mathfrak{g}} \quad (1.17)$$

for $y, z \in \mathfrak{g}$. Thus

$$\begin{aligned} (s^*, t^*) &= (\iota(s)\mu, \iota(t)\mu) \\ &= (\mu, \varepsilon(s)\iota(t)\mu) \end{aligned} \quad (1.18)$$

But then using (1.17) and the fact that $\varepsilon(y)\mu = 0$ for any $y \in \mathfrak{g}$, one has

$$(\mu, \varepsilon(s)\iota(t)\mu) = \sum_{j=0}^{m-1} (-1)^j (y_m, z_{m-j}) (\mu, \varepsilon(y_1) \cdots \varepsilon(y_{m-1}) \iota(z_m) \cdots \iota(\widehat{z_{m-j}}) \cdots \iota(z_1)\mu).$$

But then by induction and the expansion of the determinant defined by the last row one has

$$\begin{aligned} (\mu, \varepsilon(s)\iota(t)\mu) &= \det(y_i, z_j)(\mu, \mu) \\ &= (s, t) \end{aligned}$$

proving (1.14). QED

1.2. The algebra $S(\mathfrak{g})$ is a G -module extending the adjoint representation. Let $J = S(\mathfrak{g})^G$ be the subalgebra of \mathfrak{g} -invariants. Let $H \subset S(\mathfrak{g})$ be the graded \mathfrak{g} -submodule of harmonic elements in $S(\mathfrak{g})$ (See §1.4 in [K2] for definitions). Then one knows

$$S(\mathfrak{g}) = J \otimes H. \quad (1.19)$$

See (1.4.3) in [K2].

Let r be as in (1.1). For the convenience of the reader we repeat a paragraph in §1.2 of [K4]. Let $\Sigma_{2r,2}$ be the subgroup of all $\sigma \in \Sigma_{2r}$ such that σ permutes the set of

unordered pairs $\{(1, 2), (3, 4), \dots, (2r - 1, 2r)\}$. It is clear that $\Sigma_{2r,2}$ has order $r! 2^r$. Now let Π_r be a cross-section of the set of left cosets of $\Sigma_{2r,2}$ in Σ_{2r} . Thus one has a disjoint

$$\Sigma_{2r} = \bigcup_{\nu \in \Pi_r} \nu \Sigma_{2r,2}. \quad (1.20)$$

One notes that the cardinality of Π_r is $(2r - 1)(2r - 3) \cdots 1$ (the index of $\Sigma_{2r,2}$ in Σ_{2r}) and the correspondence

$$\nu \mapsto ((\nu(1), \nu(2)), (\nu(3), \nu(4)), \dots, (\nu(2r - 1), \nu(2r))) \quad (1.21)$$

sets up a bijection of Π_r with the set of all partitions of $(1, 2, \dots, 2r)$ into a union of subsets, each of which has two elements. Furthermore, since the signum character restricted to $\Sigma_{2r,2}$ is nontrivial we may choose Π_r so that

$$sg(\nu) = 1$$

for all $\nu \in \Pi_r$.

In [K4] we defined a map $\Gamma : \wedge^{2r} \mathfrak{g} \rightarrow S(\mathfrak{g})$; (Its significance will become apparent later). Here, using Proposition 1.2 in [K4] we will give a simpler definition of Γ . By Proposition 1.2 in [K4] one has

Proposition 1.2. *There exists a map*

$$\Gamma : \wedge^{2r} \mathfrak{g} \rightarrow S^r(\mathfrak{g}) \quad (1.21a)$$

such that for any $w_i \in \mathfrak{g}$, $i = 1, \dots, 2r$, one has

$$\Gamma(w_1 \wedge \cdots \wedge w_{2r}) = \sum_{\nu \in \Pi_r} [w_{\nu(1)}, w_{\nu(2)}] \cdots [w_{\nu(2r-1)}, w_{\nu(2r)}]. \quad (1.22)$$

As a polynomial function of degree r on \mathfrak{g} , one notes that

$$\Gamma(w_1 \wedge \cdots \wedge w_{2r})(x) = \sum_{\nu \in \Pi_r} ([w_{\nu(1)}, w_{\nu(2)}], x) \cdots ([w_{\nu(2r-1)}, w_{\nu(2r)}], x). \quad (1.23)$$

This clear from (1.1.7) in [K4] and (1.3) here.

The algebra $\wedge \mathfrak{g}$ is a natural G -module by extension of the adjoint representation. It is clear that Γ is a G -map. Let $M \subset S^r(\mathfrak{g})$ be the image of Γ . The following is proved as Corollary 3.3 in [K4].

Theorem 1.3. *One has $M \subset H^r$ so that M is a G -module of harmonic polynomials of degree r on \mathfrak{g} .*

Giving properties of M and determining its rather striking \mathfrak{g} -module structure is the main goal of this paper.

For any $y \in \mathfrak{g}$ one has the familiar supercommutation formula $\iota(y)d + d\iota(y) = \theta(y)$. See e.g., (92) in [K5]. Now let $x, y \in \mathfrak{g}$. Since $d\iota(y)(x) = 0$ one has $\iota(y)dx = [y, x]$. Thus, by (1.8), using superscript notation for centralizers one has

$$\text{Rad } dx = \mathfrak{g}^x. \quad (1.24)$$

Clearly $[x, \mathfrak{g}]$ is the \mathcal{B} orthogonal subspace in \mathfrak{g} to \mathfrak{g}^x so that

$$[x, \mathfrak{g}] = (\text{Rad } dx)^\perp \quad (1.25)$$

for any $x \in \mathfrak{g}$.

For any $x \in \mathfrak{g}$ one knows $\dim \mathfrak{g}^x \geq \ell$. Recall that an element $x \in \mathfrak{g}$ is called regular if $\dim \mathfrak{g}^x = \ell$. The set $\text{Reg } \mathfrak{g}$ of regular elements is nonempty and Zariski open. Its complement, $\text{Sing } \mathfrak{g}$, is the Zariski closed set of singular elements. One notes, by (1.11), that

$$\text{Sing } \mathfrak{g} = \{x \in \mathfrak{g} \mid (dx)^r = 0\}. \quad (1.26)$$

Now $\wedge^{\text{even}} \mathfrak{g}$ is a commutative algebra and hence there exists a homomorphism

$$\gamma : S(\mathfrak{g}) \rightarrow \wedge^{\text{even}} \mathfrak{g} \quad (1.27)$$

such that for $x \in \mathfrak{g}$,

$$\gamma(x) = -dx.$$

Let γ_r be the restriction of γ to $S^r(\mathfrak{g})$. The following result, established as Theorem 1.4 in [K4], asserts that Γ is the transpose of γ_r .

Theorem 1.4. *Let $y_1, \dots, y_r \in \mathfrak{g}$ and let $\zeta \in \wedge^{2r}(\mathfrak{g})$. Then*

$$(y_1 \cdots y_r, \Gamma(\zeta)) = (-1)^r (dy_1 \wedge \cdots \wedge dy_r, \zeta). \quad (1.28)$$

Now one knows that $S^r(\mathfrak{g})$ is (polarization) spanned by all powers x^r for $x \in \mathfrak{g}$. Using (1.3), (1.26) and Theorem 1.4 we recover Proposition 3.2 in [K4]. The key point is that M defines the variety $\text{Sing } \mathfrak{g}$.

Theorem 1.5. *Let $x \in \mathfrak{g}$ and $\zeta \in \wedge^{2r} \mathfrak{g}$. Then*

$$\Gamma(\zeta)(x) = \frac{(-1)^r}{r!} ((dx)^r, \zeta). \quad (1.29)$$

In particular

$$f(x) = 0, \forall f \in M \iff x \in \text{Sing}(\mathfrak{g}). \quad (1.30)$$

If \mathfrak{a} is a Cartan subalgebra of \mathfrak{g} , then one knows that $\mathfrak{a} \cap \text{Sing } \mathfrak{g}$ is a union of the root hyperplanes in \mathfrak{a} . Hence as a corollary of Theorem 1.5 one has

Theorem 1.6. *Let \mathfrak{a} be a Cartan subalgebra of \mathfrak{g} . Let $\Delta_+(\mathfrak{a})$ be a choice of positive roots for the pair $(\mathfrak{a}, \mathfrak{g})$. Then for any $f \in M$ one has*

$$f|_{\mathfrak{a}} \in \mathbb{C} \prod_{\beta \in \Delta_+(\mathfrak{a})} \beta. \quad (1.31)$$

Going to the opposite extreme we recall that a nilpotent element e is called principal if it is regular. Let e be a principal nilpotent element. Then by Corollary 5.6

in [K1] there exists a unique nilpotent radical \mathfrak{n} of a Borel subalgebra such that $e \in \mathfrak{n}$. Furthermore $\mathfrak{g}^e \cap [\mathfrak{n}, \mathfrak{n}]$ is a linear hyperplane in \mathfrak{g}^e and $\mathfrak{g}^e \cap [\mathfrak{n}, \mathfrak{n}] = (\text{Sing } \mathfrak{g}) \cap \mathfrak{g}^e$ by Theorem 5.3 and Theorem 6.7 in [K1]. Thus there exists a nonzero linear functional ξ on \mathfrak{g}^e such that

$$\text{Ker } \xi = (\text{Sing } \mathfrak{g}) \cap \mathfrak{g}^e. \quad (1.32)$$

This establishes

Theorem 1.7. *Let $e \in \mathfrak{g}$ be principal nilpotent. Let $f \in M$. Then using the notation of (1.32) one has*

$$f|_{\mathfrak{g}^e} \in \mathbb{C} \xi^r. \quad (1.33)$$

Since $\text{Sing } \mathfrak{g}$ is clearly a cone it follows that the ideal \mathcal{L} of $f \in S(\mathfrak{g})$ which vanishes on $\text{Sing } \mathfrak{g}$ is graded. One of course has that $M \subset \mathcal{L}^r$. We now observe that r is the minimal value of k such that $\mathcal{L}^k \neq 0$

Proposition 1.8. *Assume that $0 \neq f \in \mathcal{L}^k$. Then $k \geq r$.*

Proof. Since $f \neq 0$ there clearly exists a Cartan subalgebra \mathfrak{a} of \mathfrak{g} such that $f|_{\mathfrak{a}} \neq 0$. But then using the notation of Theorem 1.6 it follows from the prime decomposition that β divides $f|_{\mathfrak{a}}$ for all $\beta \in \Delta_+(\mathfrak{a})$. Thus $k \geq r$. QED

2. The structure of M in terms of minors and as a G -module

2.1. For any $z \in \mathfrak{g}$ let ∂_z be the partial derivative of $S(\mathfrak{g})$ defined by z . Let $W(\mathfrak{g}) = S(\mathfrak{g}) \otimes \wedge \mathfrak{g}$ so that $W(\mathfrak{g})$ can be regarded as the supercommutative algebra of all differential forms on \mathfrak{g} with polynomial coefficients. To avoid confusion with the already defined d , let d_W be the operator of exterior differentiation on $W(\mathfrak{g})$. That is, d_W is a derivation of degree 1 defined so that if $\{z_i, w_j\}$, $i, j = 1, \dots, n$, are dual \mathcal{B}

bases of \mathfrak{g} , then

$$d_W(f \otimes u) = \sum_i^n \partial_{z_i} f \otimes \varepsilon(w_i) u \quad (2.1)$$

where $f \in S(\mathfrak{g})$ and $u \in \wedge \mathfrak{g}$. Of course d_W is independent of the choice of bases. In particular $d_W f$ is a differential form of degree 1 on \mathfrak{g} .

For any $x \in \mathfrak{g}$ one has a homomorphism

$$W(\mathfrak{g}) \rightarrow \wedge \mathfrak{g}, \quad \varphi \mapsto \varphi(x) \quad (2.2)$$

defined so that if $\varphi = f \otimes u$, using the notation of (2.1), then $\varphi(x) = f(x)u$. Next one notes that the G -module structures on $S(\mathfrak{g})$ and $\wedge \mathfrak{g}$ define, by tensor product, a G -module structure on $W(\mathfrak{g})$. Clearly d_W is a G map. If $a \in G$ and $\varphi \in W(\mathfrak{g})$, the action of a on φ will simply be denoted by $a \cdot \varphi$. If $x \in \mathfrak{g}$ one readily has

$$a \cdot (\varphi(x)) = a \cdot \varphi(a \cdot x). \quad (2.3)$$

One knows (Chevalley) that J is a polynomial ring $\mathbb{C}[p_1, \dots, p_\ell]$ where the p_j are homogeneous polynomials. If $d_j = \deg p_j$, for $j = 1, \dots, \ell$, and $m_j = d_j - 1$, then the m_j are exponents of \mathfrak{g} so that

$$\sum_{j=1}^{\ell} m_j = r. \quad (2.4)$$

Moreover we can choose the p_j so that $\partial_y p_j \in H$ for any $y \in \mathfrak{g}$ (see Theorem 67 in [K5]). In fact, if H_{ad} is the primary component of H corresponding to the adjoint representation, then the multiplicity of the adjoint representation in H_{ad} is equal to ℓ and τ_j , $j = 1, \dots, \ell$, is a basis of $\text{Hom}_G(\mathfrak{g}, H_{\text{ad}})$ where

$$\tau_j(y) = \partial_y p_j \quad (2.5)$$

for any $y \in \mathfrak{g}$. Again see Theorem 67 in [K5].

Remark 2.2. Using the notation of (2.1) note that

$$\{w_{i_1} \wedge \dots \wedge w_{i_\ell} \mid 1 \leq i_1 < \dots < i_\ell \leq n\}$$

is a basis of $\wedge^\ell \mathfrak{g}$. Furthermore

$$\{z_{j_1} \wedge \cdots \wedge z_{j_\ell} \mid 1 \leq j_1 < \cdots < j_\ell \leq n\}$$

is the dual basis since clearly

$$(w_{i_1} \wedge \cdots \wedge w_{i_\ell}, z_{j_1} \wedge \cdots \wedge z_{j_\ell}) = \prod_{k=1}^n \delta_{i_k j_k}. \quad (2.6)$$

In addition if the w_i are a \mathcal{B} -orthonormal basis of \mathfrak{g} , then $w_i = z_i$, $i = 1, \dots, n$, and hence (2.6) implies that $\{w_{i_1} \wedge \cdots \wedge w_{i_\ell} \mid 1 \leq i_1 < \cdots < i_\ell \leq n\}$ is a \mathcal{B} orthonormal basis of $\wedge^\ell \mathfrak{g}$.

Now for any $y_i \in \mathfrak{g}$, $i = 1, \dots, \ell$, let $\psi(y_1, \dots, y_\ell) = \det \partial_{y_i} p_j$ so that

$$\psi(y_1, \dots, y_\ell) \in S^r(\mathfrak{g}) \quad (2.7)$$

by (2.4). But now $d_W p_j$ is an invariant 1-form on \mathfrak{g} . If $x \in \mathfrak{g}$, then $d_W p_j(x) \in \wedge^1 \mathfrak{g}$. Explicitly, using the notation in (2.1), one has

$$d_W p_j(x) = \sum_{i=1}^n \partial_{z_i} p_j(x) w_i. \quad (2.8)$$

One notes that $\partial_{z_i} p_j$ is an $n \times \ell$ matrix of polynomial functions. There are $\binom{n}{\ell}$ $\ell \times \ell$ minors for this matrix. The determinants of these minors all lie in $S^r(\mathfrak{g})$ and appear in the following expansion.

Proposition 2.1. *Let the notation be as in (2.1). Let $x \in \mathfrak{g}$. Then in $\wedge^\ell \mathfrak{g}$ one has*

$$d_W p_1(x) \wedge \cdots \wedge d_W p_\ell(x) = \sum_{1 \leq i_1 < \cdots < i_\ell \leq n} \psi(z_{i_1}, \dots, z_{i_\ell})(x) w_{i_1} \wedge \cdots \wedge w_{i_\ell}. \quad (2.9)$$

Proof. This is just standard exterior algebra calculus using (2.8). QED

Theorem 2.2. *Let $v_i, i = 1, \dots, n$, be a \mathcal{B} orthonormal basis of \mathfrak{g} chosen and ordered so that $v_i, i = 1, \dots, \ell$, is a basis of \mathfrak{h} . Then there exists a scalar $\kappa \in \mathbb{C}^\times$ such that, for any $y \in \mathfrak{h}$,*

$$d_W p_1(y) \wedge \cdots \wedge d_W p_\ell(y) = \kappa \left(\prod_{\varphi \in \Delta_+} \varphi(y) \right) v_1 \wedge \cdots \wedge v_\ell. \quad (2.10)$$

Proof. If $a \in G$, $x \in \mathfrak{g}$ and $j = 1, \dots, \ell$, then since $d_W p_j$ is G -invariant one has

$$a \cdot d_W p_j(x) = d_W p_j(a \cdot x). \quad (2.11)$$

But this implies that

$$d_W p_j(x) \in \text{cent } \mathfrak{g}^x \quad (2.12)$$

since if we choose $a \in G^x$ in (2.11) it follows from (2.11) that $d_W p_j(x)$ commutes with \mathfrak{g}^x . But $x \in \mathfrak{g}^x$ so that $d_W p_j(x) \in \mathfrak{g}^x$. This establishes (2.12).

Now by Theorem 9, p. 382 in [K2] one has that if $x \in \mathfrak{g}$, then

$$\{d_W p_1(x), \dots, d_W p_\ell(x)\} \text{ are linearly independent} \iff x \in \text{Reg } \mathfrak{g}. \quad (2.12a)$$

Thus the left side of (2.10) vanishes if and only if $y \in \text{Sing } \mathfrak{g} \cap \mathfrak{h}$. In particular, choosing the z_i in (2.9) so that $v_j = z_j$ for $j = 1, \dots, \ell$, one has $\psi(v_1, \dots, v_\ell)(y) = 0$ if y is singular by the expansion (2.9). On the other hand, if $y \in \mathfrak{h}$ is regular then, by (2.12), one must have that

$$\{d_W p_j(y), j = 1, \dots, \ell\} \text{ is a basis of } \mathfrak{h}. \quad (2.13)$$

Thus if y is regular, the left side of (2.10) equals $\nu v_1 \wedge \cdots \wedge v_\ell$ for some $\nu \in \mathbb{C}^\times$. Comparing with the expansion (2.9) one must have $\nu = \psi(v_1, \dots, v_\ell)(y)$. But then $\psi(v_1, \dots, v_\ell)|_{\mathfrak{h}}$ is a polynomial of degree r which vanishes on $y \in \mathfrak{h}$ if and only if $y \in \mathfrak{h}$ is singular. Thus

$$\psi(v_1, \dots, v_\ell)|_{\mathfrak{h}} = \kappa \prod_{\varphi \in \Delta_+} \varphi$$

for some nonzero constant κ . This proves (2.10). QED

2.2. For any root $\varphi \in \Delta$ let $e_\varphi \in \mathfrak{g}$ be a corresponding root vector. We will make choices so that

$$(e_\varphi, e_{-\varphi}) = 1. \quad (2.14)$$

For any $x \in \mathfrak{h}$, one then has

$$dx = \sum_{\varphi \in \Delta_+} \varphi(x) e_\varphi \wedge e_{-\varphi}. \quad (2.15)$$

See Proposition 37, p. 311 in [K5], noting (106), p. 302 and (142), p. 309 in [K5]. But then recalling (1.27) one has

$$\gamma_r(x^r) = r!(-1)^r \prod_{\varphi \in \Delta_+} \varphi(x) e_\varphi \wedge e_{-\varphi}. \quad (2.16)$$

But since $(e_\varphi \wedge e_{-\varphi}, e_\varphi \wedge e_{-\varphi}) = -1$, by (2.14), for any $\varphi \in \Delta_+$ one has that

$$\left(\prod_{\varphi \in \Delta_+} e_\varphi \wedge e_{-\varphi}, \prod_{\varphi \in \Delta_+} e_\varphi \wedge e_{-\varphi} \right) = (-1)^r. \quad (2.17)$$

But then if $\{v_i \mid i = 1, \dots, \ell\}$ is an orthonormal basis of \mathfrak{h} , one has

$$(v_1 \wedge \dots \wedge v_\ell \wedge \prod_{\varphi \in \Delta_+} e_\varphi \wedge e_{-\varphi}, v_1 \wedge \dots \wedge v_\ell \wedge \prod_{\varphi \in \Delta_+} e_\varphi \wedge e_{-\varphi}) = (-1)^r. \quad (2.18)$$

But then we may choose an ordering of the v_i such that

$$\mu = i^r v_1 \wedge \dots \wedge v_\ell \wedge \prod_{\varphi \in \Delta_+} e_\varphi \wedge e_{-\varphi} \quad (2.19)$$

so that

$$(v_1 \wedge \dots \wedge v_\ell)^* = i^r \prod_{\varphi \in \Delta_+} e_\varphi \wedge e_{-\varphi}. \quad (2.20)$$

But then one has

Theorem 2.3 *There exists $\kappa_o \in \mathbb{C}^\times$ such that for any $x \in \mathfrak{g}$,*

$$\begin{aligned} (d_W p_1(x) \wedge \dots \wedge d_W p_\ell(x))^* &= \kappa_o \frac{(-dx)^r}{r!} \\ &= \kappa_o \gamma_r\left(\frac{x^r}{r!}\right). \end{aligned} \quad (2.21)$$

Proof. If $y \in \mathfrak{h}$ is regular, then (2.21), for $y = x$, follows from (2.16),(2.20) and Theorem 2.2. That is

$$\begin{aligned} (d_W p_1(y) \wedge \cdots \wedge d_W p_\ell(y))^* &= \kappa_o \frac{(-dy)^r}{r!} \\ &= \kappa_o \gamma_r\left(\frac{y^r}{r!}\right). \end{aligned} \quad (2.22)$$

But now if $x \in \mathfrak{g}$ regular and semisimple there exist $a \in G$ and a regular $y \in \mathfrak{h}$ such that $a \cdot y = x$. But now since $*$ and γ_r are clearly G -maps one has (2.21) by applying the action of a to both sides of (2.22). However the set of regular semisimple elements in \mathfrak{g} is dense (this nonempty set is Zariski open) one has (2.21) for all $x \in \mathfrak{g}$ by continuity. QED

Returning to our module M of harmonic polynomials on \mathfrak{g} of degree r it is obvious, by definition, that M is spanned by all $f \in S^r$ of the form $f = \Gamma(w_1 \wedge \cdots \wedge w_{2r})$ where the $w_i \in \mathfrak{g}$ are linearly independent. Explicitly $\Gamma(w_1 \wedge \cdots \wedge w_{2r})$ is given by (1.22). We now show that $\Gamma(w_1 \wedge \cdots \wedge w_{2r})$ may also be given as the determinant of one of the $\ell \times \ell$ minors in the expansion (2.9).

Theorem 2.4. *Let $w_k \in \mathfrak{g}$, $k = 1, \dots, 2r$, be linearly independent and let $\mathfrak{s} \subset \mathfrak{g}$ be the span of the w_k and let $u_i \in \mathfrak{g}$, $i = 1, \dots, \ell$, be a basis of \mathfrak{s}^\perp . Then there exists a constant $\kappa_1 \in \mathbb{C}^\times$ such that*

$$\begin{aligned} \Gamma(w_1 \wedge \cdots \wedge w_{2r}) &= \kappa_1 \psi(u_1, \dots, u_\ell) \\ &= \kappa_1 \det \partial_{u_i} p_j. \end{aligned} \quad (2.23)$$

Furthermore M is the span of all $\ell \times \ell$ determinant minors $\psi(v_1, \dots, v_\ell)$ where $v_i \in \mathfrak{g}$, $i = 1, \dots, \ell$, are linearly independent.

Proof. Clearly we may choose the two dual bases in (2.1) so that the given w_k are the first $2r$ -elements of the w basis and the u_i are the last ℓ elements of the z basis. Thus there exists $\kappa_2 \in \mathbb{C}^\times$ such that

$$(u_1 \wedge \cdots \wedge u_\ell)^* = \kappa_2 w_1 \wedge \cdots \wedge w_{2r}. \quad (2.24)$$

Now let $x \in \mathfrak{g}$. Then by the expansion (2.9) one has

$$(d_W p_1(x) \wedge \cdots \wedge d_W p_\ell(x), u_1 \wedge \cdots \wedge u_\ell) = \psi(u_1, \dots, u_\ell)(x). \quad (2.25)$$

But then by (1.3), (1.14), (1.29) and (2.21) one has

$$\begin{aligned} \psi(u_1, \dots, u_\ell)(x) &= ((d_W p_1(x) \wedge \cdots \wedge d_W p_\ell(x))^*, (u_1 \wedge \cdots \wedge u_\ell)^*) \\ &= \kappa_o \kappa_2 \left(\gamma_r \left(\frac{x^r}{r!} \right), w_1 \wedge \cdots \wedge w_{2r} \right) \\ &= \kappa_1^{-1} \Gamma(w_1 \wedge \cdots \wedge w_{2r})(x) \end{aligned} \quad (2.26)$$

where $\kappa_1^{-1} = \kappa_o \kappa_2$. The last statement in the theorem is obvious since clearly u_i , $i = 1, \dots, \ell$, is an arbitrary set of ℓ -independent elements in \mathfrak{g} . QED

2.3. Let $\{z_i, w_j\}$ be the arbitrary dual bases of \mathfrak{g} as in (1.4). Then, independent of the choice of bases, the Casimir operator Cas on $\wedge^\mathfrak{g}$ is given by

$$\text{Cas} = \sum_{i=1}^n \theta(z_i) \theta(w_i).$$

We recall special cases of some results in [K3]. Let $A_\ell \subset \wedge^\ell \mathfrak{g}$ be the span in $\wedge^\ell \mathfrak{g}$ of all $[\mathfrak{c}]$ where $\mathfrak{c} \in \mathfrak{g}$ is a commutative Lie subalgebra of dimension ℓ . Since the set of such subalgebras includes, for example, Cartan subalgebras it is obvious that $A_\ell \neq 0$. In fact note that

$$[\mathfrak{g}^y] \subset A_\ell \quad (2.27)$$

for any $y \in \text{Reg } \mathfrak{g}$ since, as one knows, \mathfrak{g}^y is abelian if y is regular. Clearly A_ℓ is a G submodule of \wedge^ℓ . On the other hand, let m_ℓ be the maximal value of Cas on \wedge^ℓ and let M_ℓ be the corresponding Cas eigenspace. Again, clearly M_ℓ is a G -submodule of $\wedge^\ell \mathfrak{g}$. From the definition of M_ℓ it is obvious that $\text{Hom}_G(M_\ell, \wedge^\ell \mathfrak{g}/M_\ell) = 0$. Since $\mathcal{B}| \wedge^\ell \mathfrak{g}$ is nonsingular it follows that

$$\mathcal{B}|M_\ell \text{ is nonsingular} \quad (2.28)$$

and hence M_ℓ is self-contragredient. Noting the 1/2 in (2.1.7) of [K3] the following result is a special case of Theorem (5), p. 156 in [K3].

Theorem 2.5. *One has*

$$A_\ell = M_\ell \tag{2.29}$$

and in addition

$$m_\ell = \ell. \tag{2.30}$$

For any ordered subset $\Phi \subset \Delta$, $\Phi = \{\varphi_1, \dots, \varphi_k\}$, let $e_\Phi = e_{\varphi_1} \wedge \dots \wedge e_{\varphi_k}$ and put $\langle \Phi \rangle = \sum_{\varphi \in \Phi} \varphi$ so that with respect to \mathfrak{h} ,

$$e_\Phi \in \wedge^k \mathfrak{g} \text{ is a weight vector of weight } \langle \Phi \rangle. \tag{2.31}$$

Let $\mathfrak{b} \subset \mathfrak{g}$ be the Borel subalgebra of \mathfrak{g} spanned by \mathfrak{h} and $\{e_\varphi\}$, for $\varphi \in \Delta_+$, and put $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. Any ideal \mathfrak{a} of \mathfrak{b} where $\mathfrak{a} \subset \mathfrak{n}$ is necessarily spanned by root vectors. We will say that Φ , as above, is an ideal of Δ_+ if $\Phi \subset \Delta_+$ and $\mathfrak{a}_\Phi = \sum_{i=1}^k \mathbb{C}e_{\varphi_i}$ is an ideal in \mathfrak{b} .

Remark 2.6. One notes that if Φ is an ideal of Δ_+ and $V_\Phi \subset \wedge^k \mathfrak{g}$ is the G -module spanned by $G \cdot e_\Phi$, then V_Φ is irreducible having e_Φ as highest weight vector and $\langle \Phi \rangle$ as highest weight.

As already noted in [K3] (see bottom of p. 158) it is immediate that if \mathfrak{a} is any abelian ideal in \mathfrak{b} , then $\mathfrak{a} \subset \mathfrak{n}$ so that $\mathfrak{a} = \mathfrak{a}_\Phi$ for an ideal $\Phi \subset \Delta_+$. Much more subtly it has been established in [KW] (see Lemma 12, p. 113 in [KW]) that any ideal \mathfrak{a} of \mathfrak{b} having dimension ℓ is in fact abelian. Let \mathcal{I} be the (obviously finite) set of all ideals Φ in Δ_+ which have cardinality ℓ . If $\Phi_1, \Phi_2 \in \mathcal{I}$ are distinct, then $\langle \Phi_1 \rangle \neq \langle \Phi_2 \rangle$ by Theorem (7), p. 158 in [K3] so that V_{Φ_1} are inequivalent \mathfrak{g} and G modules. Then Theorem (8), p. 159 in [K3] implies

Theorem 2.7. M_ℓ is a multiplicity one G -module. In fact

$$M_\ell = \bigoplus_{\Phi \in \mathcal{I}} V_\Phi \quad (2.32)$$

so the number of irreducible components in M_ℓ is the cardinality of \mathcal{I} .

Remark 2.8. In the general case we do not have a formula for $\text{card } \mathcal{I}$ although computing this number in any given case does not seem to be too difficult. In the special case where $\mathfrak{g} \cong \text{Lie } Sl(n, \mathbb{C})$ one easily has a bijective correspondence of \mathcal{I} with the set of all Young tableaux of size $n - 1$ so that in this case

$$\text{card } \mathcal{I} = p(n - 1) \quad (2.33)$$

where p here is the classical partition function.

Let

$$\tau : \wedge^\ell \mathfrak{g} \rightarrow \wedge^{2r} \mathfrak{g} \quad (2.34)$$

be the G -isomorphism defined by putting $\tau(u) = u^*$ recalling that $u^* = \iota(u)\mu$. Let $M_{2r} = \tau(M_\ell)$,

Theorem 2.9. τ is a \mathcal{B} -isomorphism so that $\mathcal{B}|_{M_{2r}}$ is nonsingular. Furthermore ℓ is the maximal eigenvalue of Cas on $\wedge^{2r} \mathfrak{g}$ and M_{2r} is the corresponding eigenspace. As G modules one has

$$M_\ell \cong M_{2r} \quad (2.35)$$

so that M_{2r} is a multiplicity 1 module where in fact

$$M_{2r} \cong \bigoplus_{\Phi \in \mathcal{I}} V_\Phi. \quad (2.36)$$

We recall the V_Φ is an irreducible G -module with highest weight $\langle \Phi \rangle$. See (2.31).

Proof. The first statement follows from Proposition 1.1. The remaining statements are immediate from Theorem 2.7 since τ is a G -isomorphism. QED

In light of equality $M_\ell = A_\ell$ (see (2.9)) Rane Brylinski in her thesis (see [RB]) proved that M_ℓ is the span of $G \cdot [\mathfrak{h}]$. The thesis however has not been published. A stronger theorem (motivated by her result) appears in [KW]. The following result is just Corollary 2, p. 105 in [KW].

Theorem 2.10. *M_ℓ is the span of $G \cdot [\mathfrak{g}^x]$ for any $x \in \text{Reg } \mathfrak{g}$.*

Now by (2.12) and (2.12a) one has

$$\mathbb{C} d_W p_1(x) \wedge \cdots \wedge d_W p_\ell(x) = [\mathfrak{g}^x] \quad (2.37)$$

for any $x \in \text{Reg } \mathfrak{g}$. Using Theorem 2.3 we can now transfer Theorem 2.10 to M_{2r} where it will have consequences for the structure of the space of functions $M \subset H^r$.

Theorem 2.11. *M_{2r} is the span of $G \cdot (\gamma_r(\frac{x^r}{r!}))$ for any $x \in \text{Reg } \mathfrak{g}$.*

Proof. This is immediate from Theorem 2.3, Theorem 2.10, (2.37) and the fact that τ is a G -isomorphism. QED.

Let N_{2r} be the \mathcal{B} orthogonal subspace to M_{2r} in $\wedge^{2r} \mathfrak{g}$. By the first statement in Theorem 2.9 one has a \mathcal{B} orthogonal G -module decomposition $\wedge^{2r} \mathfrak{g}$,

$$\wedge^{2r} \mathfrak{g} = N_{2r} \oplus M_{2r}. \quad (2.38)$$

Remark 2.12. Note that by Theorem 2.9 any eigenvalue of Cas in N_{2r} is less than ℓ .

We return now to our G -space M of homogeneous harmonic polynomials on \mathfrak{g} of degree r which define $\text{Sing } \mathfrak{g}$. We recapitulate some of the properties of $M = \Gamma(\wedge^{2r} \mathfrak{g})$ already established in this paper. Let $w_k \in \mathfrak{g}$, $k = 1, \dots, 2r$, be linearly independent and let $z_i \in \mathfrak{g}$, $i = 1, \dots, \ell$, be linearly independent and \mathcal{B} orthogonal to the w_k . Then

for suitable generators p_j , $j = 1, \dots, \ell$, of $J = S(\mathfrak{g})^G$, we have

- (1) $\Gamma(w_i \wedge \dots \wedge w_{2r})$ is explicitly given by (1.23)
- (2) $\Gamma(w_i \wedge \dots \wedge w_{2r})$ is given as (up to scalar multiplication) $\det \partial_{z_i} p_j$. See Theorem 2.4.
- (3) If $f \in M$, then $f|_{\mathfrak{a}}$, where \mathfrak{a} is any Cartan subalgebra or $\mathfrak{a} = \mathfrak{g}^e$ for e principal nilpotent, is given in Theorems 1.6 and 1.7.

We now determine the G -module structure of M ,

Theorem 2.13. $N_{2r} = \text{Ker } \Gamma$ and

$$\Gamma : M_{2r} \rightarrow M \quad (2.39)$$

is a G -isomorphism so that as G -modules

$$M \cong M_{2r} \cong M_\ell = A_\ell \quad (2.40)$$

where we recall $A_\ell \subset \wedge^\ell \mathfrak{g}$ has been defined in [K3] as the span of $[\mathfrak{s}]$ over all abelian subalgebras $\mathfrak{s} \subset \mathfrak{g}$ of dimension ℓ .

Furthermore we have defined \mathcal{I} as the set of all ideals Φ in Δ_+ of cardinality ℓ , parameterizing with the notation \mathfrak{a}_Φ , the set of all ideals \mathfrak{a} of \mathfrak{b} having dimension ℓ . See Remark 2.6.

Moreover M is a multiplicity one G -module with $\text{card } \mathcal{I}$ irreducible components. In addition \mathcal{I} parameterizes these components in the sense that the component corresponding to $\Phi \in \mathcal{I}$ is equivalent to V_Φ , using the notation of Remark 2.6, and hence has highest weight $\langle \Phi \rangle$. Finally Cas takes the value ℓ on each and every irreducible component of M .

Proof. By (1.27) and (1.29) one has

$$(\Gamma(\zeta)(x) = (\zeta, \gamma_r(\frac{x^r}{r!})) \quad (2.41)$$

for any $x \in \mathfrak{g}$ and any $\zeta \in \wedge^{2r} \mathfrak{g}$. Of course $\gamma_r(\frac{x^r}{r!}) = 0$ for any $x \in \text{Sing } \mathfrak{g}$ (see (2.12a) and Theorem 2.3). However M_{2r} is the span of $G \cdot \gamma_r(\frac{x^r}{r!})$ for any $x \in \text{Reg } \mathfrak{g}$ by Theorem 2.11. Thus not only does (2.41) imply that $N_{2r} \subset \text{Ker } \Gamma$ but $N_{2r} = \text{Ker } \Gamma$ since if $\zeta \in M_{2r}$ and $x \in \text{Reg } \mathfrak{g}$ there exists $a \in G$ such that if $y = a \cdot x$, then $\Gamma(\zeta)(y) \neq 0$ by Theorem 2.11 and the nonsingularity of $\mathcal{B}|_{M_{2r}}$, as asserted in Theorem 2.9. Since Γ is a G -map one has the isomorphism (2.39). The remaining statements follow from Theorem 2.5 and Theorem 2.9. QED

References

- [K1] B. Kostant, The Three Dimensional Sub-Group and the Betti Numbers of a Complex Simple Lie Group, *Amer. Jour. of Math.*, **81**(1959), 973–1032.
- [K2] B. Kostant, Lie Group Representations on Polynomial Rings, *Amer. J. Math.*, **85**(1963), 327–404.
- [K3] B. Kostant, Eigenvalues of a Laplacian and Commutative Lie Subalgebras, *Topology*, **13**(1965) 147–159.
- [K4] B. Kostant, A Lie Algebra Generalization of the Amitsur-Levitski Theorem, *Adv. in Math.*, **40**(1981), No. 2, 155–175.
- [K5] B. Kostant, Clifford Algebra Analogue of the Hopf-Koszul-Samelson Theorem, the ρ -Decomposition, $C(\mathfrak{g}) = \text{End } V_\rho \otimes C(P)$, and the \mathfrak{g} -Module Structure of $\wedge \mathfrak{g}$, *Adv. in Math.*, **125**(1997), 275–350.
- [KW] B. Kostant and N. Wallach, On a theorem of Ranee Brylinski, *Contemporary Mathematics*, **490**(2009), 105–142.
- [Kz] J. L. Koszul, Homologie et cohomologie des algèbres de Lie, *Bull. Soc. Math. Fr.*, **78**(1950), 65–127.
- [RB] R. Brylinski, *Abelian algebras and adjoint orbits*, Thesis MIT, 1981.